Homework related to Lecture 1: **Conservation laws**

Start with any set of partial differential equations that have the form of evolution equations, so they define how some quantity (or set of quantities) evolves. A **conservation law** is an equation of the form

$$\partial_t T + \nabla \cdot \boldsymbol{F} = 0, \qquad (*)$$

where the scalar $T(\boldsymbol{x},t)$ is called a *density*, and the vector $\boldsymbol{F}(\boldsymbol{x},t)$ is the corresponding *flux*. Here $\boldsymbol{x}$ represents spatial position (say in 3-d), t represents time, and $(\nabla \cdot)$ is the spatial divergence operator ($\boldsymbol{i}\,\partial_x + \boldsymbol{j}\,\partial_y + \boldsymbol{k}\,\partial_z$ in 3-d). Integrate (*) over a fixed domain, $\mathcal{D}$, enclosed by a smooth surface, $\mathcal{S}$, and use the divergence theorem to obtain

$$\frac{d}{dt}\iiint_D [T]dx + \oint\!\!\oint_S [F \cdot \hat{n}]ds = 0. \qquad (**)$$

The first integral is a 3-d volume integral, the second is a 2-d surface integral, and $\hat{n}$ is the outward-facing unit normal vector. If $\oint\!\!\oint_S [F \cdot \hat{n}]ds = 0$ (perhaps because $\boldsymbol{F}\cdot\hat{n} = 0$ on $\mathcal{S}$, or because of periodic boundary conditions), then $\iiint_D [T]dx$ is a constant of the motion (*i.e.*, it is *conserved*).

For the water-wave problem, the conservation laws of most interest are z-independent, so they have the form

$$\partial_t T + \partial_x (F_1) + \partial_y (F_2) = 0.$$

Typically, these arise by integrating in z. The purpose of this homework set is to use the equations derived in Lecture 1 to derive five such conservation laws. (The entire problem set can be viewed as an exercise in integration by parts, plus Leibniz' Rule to differentiate integrals.)

1. Mass conservation is easy

Start with the continuity equation, $\nabla \cdot \boldsymbol{u} = 0$, where $\boldsymbol{u} = (u,v,w)$ is the (3-d) velocity vector in the water. Integrate in z to obtain $\int_{-h}^{\eta} [\nabla \cdot u]dz = 0$, then use the boundary conditions at $z = \eta(x,y,t)$ and at $z = -h(x,y)$.

<u>Show</u>

$$\partial_t(\eta) + \partial_x\left(\int_{-h}^{\eta}[u]dz\right) + \partial_y\left(\int_{-h}^{\eta}[v]dz\right) = 0. \qquad (1)$$

<u>In what sense does (1) prove that mass is conserved?</u>

[*Hint*: Let ρ denote the mass-density of water. Then $\iint_A [\int_{-h}^{\eta} \rho dz] dxdy$ is the total mass of a column of water above the region A in the x-y plane.]

Answers are given at the end of each problem set.

2. Energy conservation is also straightforward. Start with $(\partial_t \phi)$•(*continuity*):

$$(\partial_t \phi) \bullet (\nabla^2 \phi) = 0, \qquad -h < z < \eta(x,y,t).$$

Integrate in z and show:

$$\partial_t [\frac{1}{2} \int_{-h}^{\eta} |\nabla \phi|^2 \, dz + \frac{1}{2} g \eta^2 + \frac{\sigma}{\rho} (\sqrt{1 + |\nabla \eta|^2} - 1)] + \partial_x (F_1) + \partial_y (F_2) = 0. \tag{2}$$

What are F_1 and F_2 ?

Observe that for a water column above a region in the x-y plane with area A,

$\iint_A [\frac{\rho}{2} \int_{-h}^{\eta} |\nabla \phi|^2 dz] dxdy$ = kinetic energy in water column over A

$\iint_A [\frac{\rho g \eta^2}{2}] dxdy$ = potential energy due to gravity

$\iint_A [\sigma(\sqrt{1 + |\nabla \eta|^2} - 1)] dxdy$ = surface energy (due to surface tension).

[The extra (-1) is added so that surface energy vanishes for a flat surface.]

3. Horizontal momentum is more complicated. From here on, assume that the bottom is flat and horizontal, so $h(x,y)$ = constant. Horizontal momentum is a vector with two components. For the x-component:

a) Start with (u)•(*continuity*): $(\partial_x \phi) \cdot (\nabla^2 \phi) = 0$ on $-h < z < \eta(x,y,t)$.
Integrate in z and show:

$$\partial_x (\int_{-h}^{\eta} (\partial_x \phi)^2 dz) + \partial_y (\int_{-h}^{\eta} (\partial_x \phi \cdot \partial_y \phi) dz) - \partial_x (\int_{-h}^{\eta} \tfrac{1}{2} |\nabla \phi|^2 dz) + \partial_t \eta \cdot \partial_x \phi |_{z=\eta} + \frac{1}{2} \partial_x \eta \cdot (|\nabla \phi|^2) |_{z=\eta} = 0.$$

b) It is convenient to introduce $\psi(x,y,t)$, which also appears in Zakharov's formulation of the water-wave problem as a Hamiltonian system (in Lecture 3). Define

$$\psi(x,y,t) = \phi(x,y,z,t) |_{z=\eta(x,y,t)}$$

so that

$$\partial_x \psi = \partial_x \phi |_{z=\eta} + \partial_x \eta \cdot (\partial_z \phi |_{z=\eta})$$

and

$$\partial_t \psi = \partial_t \phi|_{z=\eta} + \partial_t \eta \cdot (\partial_z \phi|_{z=\eta}).$$

Show

$$\partial_t \eta \cdot \partial_x \phi|_{z=\eta} = \partial_x(\psi \partial_t \eta) - \partial_t(\psi \partial_x \eta) + \partial_x \eta \cdot \partial_t \phi|_{z=\eta}.$$

c) Substitute the results from (b) into those from (a), use the boundary conditions at the free surface, and show

$$\partial_t(\psi \partial_x \eta) = \partial_x(f_1) + \partial_y(f_2), \qquad (3)$$

where $f_2 = \int_{-h}^{\eta} [\partial_x \phi \cdot \partial_y \phi] dz + \frac{\sigma}{\rho} \frac{\partial_x \eta \partial_y \eta}{\sqrt{1 + |\nabla \eta|^2}}$.

What is f_1?

d) The (local) momentum in the x-direction at a point (x,y,z) is defined to be $[\rho u]$, so the vertically integrated (or total) momentum in the x-direction is

$$M_{(x)}(x,y,t) = \int_{-h}^{\eta} [\rho u] dz.$$

Show that

$$M_{(x)} = \partial_x\left(\rho \int_{-h}^{\eta} [\phi] dz\right) - \rho \cdot \psi \cdot \partial_x \eta.$$

Rewrite (3) as

$$\partial_t(M_{(x)}) + \partial_x(F_1) + \partial_y(F_2) = 0. \qquad (3.1)$$

What are F_1 and F_2 ?

4. Find the conservation law for the **y-component of momentum**, also under the assumption that h = constant. Write down two forms, corresponding to (3) and (3.1).

5. The centroid of a wave

Consider a localized disturbance (or wave), in which $|\boldsymbol{u}| \to 0$ as $(x^2 + y^2) \to \infty$, for $-h < z < \eta(x,y,t)$, and also $\eta \to 0$ as $(x^2 + y^2) \to \infty$. If both h and $|\boldsymbol{u}|$ vanish quickly enough as $(x^2 + y^2) \to \infty$, then we may define the *mass of the wave* to be

$$m = \iint [\rho \eta] dx dy. \qquad (5)$$

[*Explanation*: From problem 1, $\iint_A [\int_{-h}^{\eta} \rho dz] dx dy$ is the total mass of a column of water above the region A in the x-y plane. When the fluid is at rest and the quiet surface is at $z = 0$, the total mass is $\iint_A [\int_{-h}^{0} \rho dz] dx dy$, so ($m$) represents the "added mass" due to the wave. Note that (m) can be either positive or negative.]

For $t > 0$, the wave evolves according to the equations of (non-dissipative) water waves. According to (1) and (5), the mass of the wave (m) is conserved, but its location can change.
Show that (m) is a constant.

Then define the "centroid of the wave" as follows. Define

$$X(t) = \rho \iint [x\eta(x,y,t)]dxdy,$$
$$Y(t) = \rho \iint [y\eta(x,y,t)]dxdy.$$

Then the (horizontal) coordinates of the centroid are $\{\bar{x} = X(t)/m,\ \bar{y} = Y(t)/m\}$.
Show that

$$\frac{dX}{dt} = \text{constant}, \qquad \frac{dY}{dt} = \text{constant},$$

so the centroid moves with constant velocity. What are the components of this velocity? [*Hint*: Use problem 1.]

6. Interpretation (no work required by you): Gravity acts vertically, so there are no net horizontal forces (if the bottom is flat and horizontal, so h = const.). Therefore horizontal momentum is conserved (as shown above), and the centroid moves horizontally with a constant velocity.

Answers:

2. **Energy**: $\partial_t(Energy) + \partial_x(F_1) + \partial_y(F_2) = 0$,

where
$$F_1 = -\int_{-h}^{\eta}[\partial_t\phi\partial_x\phi]dz - \frac{\sigma}{\rho}\{\partial_t\eta\frac{\partial_x\eta}{\sqrt{1+|\nabla\eta|^2}}\},$$
$$F_2 = -\int_{-h}^{\eta}[\partial_t\phi\partial_y\phi]dz - \frac{\sigma}{\rho}\{\partial_t\eta\frac{\partial_y\eta}{\sqrt{1+|\nabla\eta|^2}}\}.$$

3. **Horizontal momentum**:

(c)
$$f_1 = \int_{-h}^{\eta}(\partial_x\phi)^2]dz - \int_{-h}^{\eta}\frac{|\nabla\phi|^2}{2}dz + \psi\partial_t\eta - \frac{g}{2}\eta^2$$
$$+\frac{\sigma}{\rho}\frac{(\partial_x\eta)^2}{\sqrt{1+|\nabla\eta|^2}} - \frac{\sigma}{\rho}\sqrt{1+|\nabla\eta|^2}.$$

(d) $F_1 = \rho f_1 - \partial_t\int_{-h}^{\eta}[\rho\phi]dz, \quad F_2 = \rho f_2.$

5. **Centroid**: $\frac{dX}{dt} = M_{(x)}, \quad \frac{dY}{dt} = M_{(y)}.$