

Homework related to Lecture 2: **Linear theory**

The linearized equations of water waves above a flat, horizontal bottom are

$$\begin{aligned}\partial_t \eta &= \partial_z \phi, \\ \partial_t \phi + g\eta &= \frac{\sigma}{\rho} \nabla^2 \eta && \text{on } z = 0, \\ \nabla^2 \phi &= 0 && \text{for } -h < z < 0, \\ \partial_z \phi &= 0 && \text{on } z = -h.\end{aligned}$$

Here we assume that g , h and $\frac{\sigma}{\rho}$ are all non-negative constants. These equations also needs suitable boundary conditions in x and in y , and they need initial conditions for $\eta(x,y)$ and $\phi(x,y,0, 0)$.

1. Conservation laws

(a) Following the procedure used in problem set 1, show that the linearized equations conserve mass:

$$\partial_t \eta + \partial_x F_1 + \partial_y F_2 = 0.$$

What are F_1 and F_2 ?

(b) Show that the linearized equations also conserve energy:

$$\partial_t E + \partial_x F_3 + \partial_y F_4 = 0,$$

where

$$E = \frac{1}{2} \left\{ g\eta^2 + \frac{\sigma}{\rho} |\nabla \eta|^2 + \int_{-h}^0 [|\nabla \phi|^2] dz \right\}.$$

What are F_3 and F_4 ?

2. Properties of one Fourier mode

We showed in class that a general solution of the linearized equations (above) on the whole x - y plane, is

$$\begin{aligned}\eta(x,y,t) &= \frac{1}{(2\pi)^2} \iint [H_-(k,l)e^{ikx+ily-i\omega t}] dkdl \\ &+ \frac{1}{(2\pi)^2} \iint [H_+(k,l)e^{ikx+ily+i\omega t}] dkdl,\end{aligned}$$

$$\phi(x, y, z, t) = \frac{1}{(2\pi)^2} \iint [\Phi_-(k, l) e^{ikx + ily - i\omega t} \frac{\cosh\{\kappa(z+h)\}}{\cosh\{\kappa h\}}] dk dl$$

$$+ \frac{1}{(2\pi)^2} \iint [\Phi_+(k, l) e^{ikx + ily + i\omega t} \frac{\cosh\{\kappa(z+h)\}}{\cosh\{\kappa h\}}] dk dl,$$

where

$$\kappa^2 = k^2 + l^2, \quad \omega^2 = (g + \frac{\sigma}{\rho} \kappa^2) \cdot \kappa \tanh\{\kappa h\}, \text{ and } (\Phi_-, H_-) \text{ are related, as}$$

are $\{\Phi_+, H_+\}$.

Here we explore the properties of a single Fourier mode. Assume that the wave in question is traveling purely in the x -direction, so $\kappa = k, l = 0$. Ocean waves with periods near 10 seconds are common. For a 10-second wave:

- What is the frequency (ω) of the wave?
- The wavenumber (k) and spatial wavelength ($2\pi/k$) depend on the parameters $\{g, h, \frac{\sigma}{\rho}\}$. For $g = 980 \text{ cm/sec}^2$ and $\frac{\sigma}{\rho} = 74 \text{ cm}^3/\text{sec}^2$, find the wavelength of the wave in water of depth: (i) 3000 m; (ii) 10 m and (iii) 2 m (where you swim).
- At what speed do wave crests move in each of these depths?
- If the horizontal velocity of the fluid ($u = \partial_x \phi$) ranges over $[-0.5 < u < 0.5]$ m/sec at the free surface ($z = 0$ in the linearized problem), find the horizontal velocity at $z = -h$, for each of these depths.

3. Particle paths

The equations for the linearized motion are written in Eulerian coordinates, but an individual fluid particle, with coordinates $(X(t), Y(t), Z(t))$, follows a path defined by

$$\frac{dX}{dt} = u(x, y, z, t) = \partial_x \phi(x, y, z, t), \quad \frac{dY}{dt} = v(x, y, z, t) = \partial_y \phi(x, y, z, t),$$

$$\frac{dZ}{dt} = w(x, y, z, t) = \partial_z \phi(x, y, z, t).$$

In this problem we find approximate particle paths. As in problem 2, assume **no velocity in the y -direction**, so it is only necessary to work out $X(t)$ and $Z(t)$.

A single Fourier mode has a velocity potential given either by

$$\phi_h(x, z, t) = \frac{v}{k} \sin(kx - \omega t) \frac{\cosh(k(z+h))}{\cosh(kh)}, \text{ or by } \phi_\infty(x, z, t) = \frac{v}{k} \sin(kx - \omega t) e^{|k|z}$$

for a wave traveling over a flat horizontal bottom (ϕ_h), or for a wave traveling on infinitely deep water (ϕ_∞); (v) represents a typical fluid velocity at the free surface. For each of the two velocity potentials above, assume that the fluid particle moves very little from its original position. In that case, we can solve

$$\frac{dX}{dt} = \partial_x \phi, \quad \frac{dZ}{dt} = \partial_z \phi,$$

assuming **no** variation of ϕ with respect to X or Z . In this way, find approximate formulae for $X(t)$ and $Z(t)$, assuming that at $t = 0$, $X(0) = x$, $Z(0) = z$.

- Show that at a fixed spatial location (*i.e.*, x , z , fixed), a fluid particle with coordinates $(X(t), Z(t))$ traces out a closed path.
- How long does the particle take to complete one loop of its circuit? Call this time T .
- Sketch the particle paths for three depths for each velocity potential: $z = -0.01h$, $z = -0.5h$, $z = -0.99h$. For each of the velocity potentials, how are the particle paths at these different depths related?
- Show that the speed of any fluid particle never exceeds $|v|$, anywhere in the fluid.

4. More on particle paths

Stokes (1847) seems to have been the first person to derive the result obtained in problem 3. Stokes also decided that this answer can't be completely right, because we observe that a strong wave field can transport mass. (For example, your favorite beach can get covered with dead seaweed by a strong wave field.) For a wave on deep water only (*i.e.*, ϕ_∞ only), this problem finds an approximate correction to the position of a fluid particle. This slow, nonperiodic flow is called "*Stokes drift*".

The error we made in problem 2 arose when we assumed that the fluid velocity depends only on (x, z, t) , rather than on $(X(t), Z(t), t)$. Therefore the correction in horizontal velocity, which we denote by U_d , is

$$U_d = u\{X(t), Z(t), t\} - u\{x, z, t\}.$$

Expanding U_d in Taylor series for small $\{X - x, Z - z\}$, and keeping only the first few terms, leads to

$$U_d = (X(t) - x) \partial_x u(x, z, t) + (Z(t) - z) \partial_z u(x, z, t) + O((X-x)^2, (Z-z)^2).$$

- Using your approximate results from problem 3, find a time-dependent result for U_d .
- Then average over time to find a mean (*i.e.*, time-independent) flow. This is the Stokes drift velocity. The linearized velocity potential, at the beginning of this assignment, generates a fluid velocity field that decreases exponentially with depth. How does the spatial variation of the Stokes drift compare with the spatial variation of the original velocity field?
- In terms of the time T , found in problem 3, how long does it take a fluid particle at the free surface to drift one spatial period of the wave ($\frac{2\pi}{k}$)?

5. Dispersion relations

Let $\omega(k)$ represent the dispersion for some energy-conserving system. For simplicity, assume that the system in question involves only one spatial dimension, so that the wavenumber, k , is a scalar. Recall the two speeds related to wave propagation in linear (or linearized) problems:

$$c_p(k) = \frac{\omega(k)}{k}, \quad c_g(k) = \frac{d\omega}{dk}.$$

a) Show that if $\omega(k)$ is differentiable, then

$$\frac{dc_p}{dk} = \frac{c_g - c_p}{k}.$$

Show from this that for $k > 0$:

- $\{c_p > c_g\}$ decreases as k increases $\Leftrightarrow \{c_p > c_g\}$;
- $c_p = c_g$ only at a critical point of $c_p(k)$,
- if the problem is nondispersive, so that $c_p = c$ for all k , then $c_p = c_g = c$.

b) (No work required by you) The magnitude of c_p relative to c_g can be observed experimentally.

- If $c_p > c_g > 0$, then wave crests move faster than a localized wave packet, so individual crests exit through the front of the packet.
- If $c_g > c_p > 0$, then wave crests move slower than a localized wave packet, so crests exit through the back of the packet.
- If $c_p = c_g$, then wave crests move at the same speed as a localized wave packet, so there exist wave packets that move as waves of permanent form.

c) The linearized dispersion relation for the water wave equations, for water subject to gravity and surface tension and resting on a horizontal bed at $z = -h$ is

$$\omega^2 = \left\{ gk + \frac{\sigma}{\rho} k^3 \right\} \tanh(kh),$$

where $\{g, \sigma/\rho, h\}$ are positive constants.

- Show that $|c_p(k)|$ has a local maximum at $k = 0$ only if $\frac{\sigma}{\rho gh^2} \geq 1/3$.

$\frac{\sigma}{\rho gh^2}$ measures the relative strength of surface tension and gravity for very long waves.

- Show that if $\frac{\sigma}{\rho gh^2} \geq 1/3$, then $|c_p|$ has a local maximum at $k = 0$, and one local minimum for $k > 0$, and a symmetric minimum for $k < 0$.
- After drawing $c_p(k)$, use your results from (a) to sketch in the curve for $c_g(k)$.

- For water of finite depth, the location (in k-space) of the minimum of $|c_p|$ is a complicated function of $\{g, \sigma/\rho, h\}$. For deep water ($kh \rightarrow \infty$), show that this minimum occurs at $\frac{\sigma k^2}{\rho g} = 1$. This minimum provides a convenient boundary, with gravity-dominated waves having smaller values of k^2 and waves controlled by surface tension having values of k^2 larger than it.

Answers:

1. a) $F_1 = -\int_{-h}^0 [u] dz = -\int_{-h}^0 [\partial_x \phi] dz, \quad F_2 = -\int_{-h}^0 [v] dz = -\int_{-h}^0 [\partial_y \phi] dz.$

$$F_3 = -\int_{-h}^0 [\partial_t \phi \partial_x \phi] dz - \frac{\sigma}{\rho} [\partial_t \eta \partial_x \eta],$$

b)

$$F_4 = -\int_{-h}^0 [\partial_t \phi \partial_y \phi] dz - \frac{\sigma}{\rho} [\partial_t \eta \partial_y \eta].$$

All of the answers in problem 1 follow from answers in problem set 1, by taking appropriate limits.

2. $\omega = 0.628 \text{ sec}^{-1}$

$L = 156 \text{ m}$ for $h = 3000 \text{ m}$, $L = 92.4 \text{ m}$ for 10 m , $L = 43.7 \text{ m}$ for 2 m .

$c_p = 15.6 \text{ m/sec}$ for $h = 3000 \text{ m}$, $c_p = 9.24 \text{ m/sec}$ for 10 m , $c_p = 4.37 \text{ m/sec}$ for 2 m .

$\nu = 10^{-52}$ for $h = 3000 \text{ m}$, $\nu = 0.40 \text{ m/sec}$ for $h = 10 \text{ m}$, $\nu = 0.48 \text{ m/sec}$ for $h = 2 \text{ m}$.

3.
$$X(t) = x + \frac{v}{\omega} [\sin(kx) - \sin(kx - \omega t)] \frac{\cosh(k(z+h))}{\cosh(kh)},$$

$$Z(t) = z - \frac{v}{\omega} [\cos(kx) - \cos(kx - \omega t)] \frac{\sinh(k(z+h))}{\cosh(kh)},$$
 for finite depth,

$$X(t) = x + \frac{v}{\omega} [\sin(kx) - \sin(kx - \omega t)] e^{|k|z},$$
 for infinite depth.

$$Z(t) = z - \frac{v}{\omega} [\cos(kx) - \cos(kx - \omega t)] e^{|k|z},$$

The time for one loop of the fluid particle is the period of the wave: $T = \frac{2\pi}{\omega}$.

4. $U_d = v^2 \left(\frac{k}{\omega}\right) e^{2|k|z} = v \left(\frac{v}{c_p}\right) e^{2|k|z}$