More homework related to Lecture 2: **The method of stationary phase**

The linearized equations of water waves with no surface tension, above a flat, horizontal bottom are

$$\partial_t \eta = \partial_z \phi, \qquad \text{on } z = 0,$$
$$\partial_t \phi + g\eta = 0$$
$$\nabla^2 \phi = 0 \qquad \text{for } -h < z < 0,$$
$$\partial_z \phi = 0 \qquad \text{on } z = -h,$$

where g and *h* are positive constants. The purpose of this homework set is to predict (approximately, for large times) the wave field that evolves from given initial data, in a 2-D problem. [No complications arise if surface tension is also included. It is omitted here to make this example problem as simple as possible.]

To do so, we use the *method of stationary phase,* originally devised by Kelvin (1887) to describe the wave pattern observed in the wake of a moving ship. The method shows clearly how the group velocity becomes important in the long-time evolution of waves that evolve in the linearized equations of water waves. This method is not discussed in the lectures because of time constraints, but it is explained in these notes (and elsewhere).

If there is no *y*-dependence, the general solution of the linearized equations (above) is

$$\eta(x,t) = \frac{1}{(2\pi)} \int [H_-(k) e^{ikx - i\omega t}] dk + \frac{1}{(2\pi)} \int [H_+(k) e^{ikx + i\omega t}] dk,$$
$$\phi(x,z,t) = \frac{1}{(2\pi)} \int [\Phi_-(k) e^{ikx - i\omega t} \frac{\cosh\{k(z+h)\}}{\cosh\{\kappa h\}}] dk$$
$$+ \frac{1}{(2\pi)} \int [\Phi_+(k) e^{ikx + i\omega t} \frac{\cosh\{k(z+h)\}}{\cosh\{\kappa h\}}] dk,$$

where

$\omega^2 = gk \tanh(kh)$, and $(\Phi_-, H_-\}$ are related, as are $\{\Phi_+, H_+\}$.

There are many ways to write the formulae for Fourier transforms. The convention used above is

$$f(x) = \frac{1}{2\pi} \int [F(k) e^{ikx}] dk, \quad F(k) = \int [f(x) e^{-ikx}] dx.$$

1. Fourier fun

a) Let

$$f(x) = a \sin^2(\frac{\pi x}{L}), \quad 0 < x < L,$$
$$f(x) = -a \sin^2(\frac{\pi x}{L}), \quad -L < x < 0,$$
$$f(x) = 0, \qquad \text{otherwise.}$$

Find $F(k)$, the Fourier transform of $f(x)$. [This is a straight-forward, somewhat tedious computation. Either work it out by hand or use a program to find $F(k)$. Either way, check your answer by evaluating $f(x)$ through the inverse transform.]
b) Sketch both $f(x)$ and $F(k)$. [$F(k)$ should be purely imaginary, so you can plot $iF(k)$.]

Answers are given at the end of this problem set.

c) Comment (no work required by you): You should find that $F(0) = 0$. This is important, both in terms of the complexity of the solution and in the corresponding physical problem.

(i) *Mathematically*: Recall from the dispersion relation that linearized water waves are dispersive for most values of k, but only weakly dispersive as $kh \to 0$. The formulae given below for stationary phase are *not* valid at $kh = 0$. But because $F(0) = 0$ for this problem, this limitation causes no difficulties. A problem with $F(0) \neq 0$ would have a separate region to consider, near $kh = 0$.

(ii) *Physically*: Recall that $F(0)$ measures the net area under the curve, $f(x)$. Hence, $F(0)$ represents the net volume (or mass) added or subtracted by the initial data. $F(0) = 0$ means that the waves generated by these initial data carry zero net volume. We will see in Lecture 5 that the net volume is very important in practical problems of water waves.

2. Set up the water wave problem

Imagine conducting an experiment on water waves of small amplitude in a long, narrow wave tank, so you can use the 2-D linearized theory, above. At $t = 0$, you carefully measure the initial shape of the free surface. You find

$$\eta(x,0) = f(x),$$

where $f(x)$ is given in problem 1. Unfortunately you forget to measure the initial velocity field (at $t = 0$), but you observe that for $t > 0$, waves move **only to the right**. From that information, find $H_-(k)$, $H_+(k)$, $\Phi_-(k)$, $\Phi_+(k)$.

The method of stationary phase

Armed with the formulae for $H_-(k)$, $H_+(k)$ found in problem 2, we can use the method of stationary phase to find an approximate description of the surface of the fluid at any large time. Next comes an explanation of the method.

Wave groups typically travel (with whatever speed they have), so if t is large, then x must also be large where the wave groups are. Set $x = vt$ in your formula for $\eta(x,t)$ from problem 2, where v is a (constant) parameter that you can choose. This change of variables replaces an integral with two free parameters $\{x,t\}$ with a related integral with two free parameters $\{v\ (=x/t), t\}$. There is no loss if information.

Your integral(s) from problem 2 should be of the form

$$\int Q(k)e^{i\{kx-\omega(k)t\}}dk = \int Q(k)e^{i\{kv-\omega(k)\}t}dk\,.$$

For $\{v.t\}$ fixed, we define the *phase* of the integral by

$$\varphi(k;v)t = \{kv - \omega(k)\}t. \qquad \textbf{(A)}$$

(First comes a hand-waving argument, to make the calculation plausible.) For large *t* (with *v,t* fixed), the phase changes rapidly from one *k* to a nearby *k*, so one would expect destructive interference from most Fourier modes, with little or no contribution to the integral. But this argument should fail for regions of *k*-space in which the phase varies slowly or not at all. Values of *k* near those points should contribute nearly the same amount to the integral, so they should interfere constructively, and make a larger contribution to the integral. Thus one is led to look for points (of *k*-space) where the *phase is stationary*,

$$\frac{d\varphi}{dk}(k;v) = \varphi'(k;v) = 0, \qquad \textbf{(B)}$$

because these points should give the dominant contributions to the integral for large *t*. (After this we write only $\varphi(k)$, because *v* is fixed.)

Now make this hand-waving argument concrete. Break the integral into pieces, so that each piece integrates over a finite band of *k*-space. For fixed $\{v,t\}$, for each of these pieces for which (**B**) is **not** true, one can integrate by parts to show that the contribution to the integral from this piece is $O(t^{-1})$:

$$\begin{aligned}\int_{K_1}^{K_2} Q(k)e^{i\varphi(k)t}dk &= \frac{-i}{t}\int_{K_1}^{K_2}[\frac{Q(k)}{\varphi'(k)}]\cdot[e^{i\varphi(k)t}i\varphi'(k)]dk \\ \frac{-i}{t}\left[\frac{Q(k)}{\varphi'(k)}e^{i\varphi(k)t}\right]_{K_1}^{K_2} &+ \frac{i}{t}\int_{K_1}^{K_2}[\frac{d}{dk}(\frac{Q(k)}{\varphi'(k)})]e^{i\varphi(k)t}dk\end{aligned} \qquad \textbf{(C)}$$

If the integrand in the new integral is absolutely integrable, then the new integral $\to 0$ as $t \to \infty$, by the Riemann-Lebesgue Lemma. Hence the contribution to the integral is $O(t^{-1})$ at most. And if the boundary contributions from neighboring pieces of the integral cancel, then the contribution is even smaller.

Consider next the pieces of the integral where (**B**) fails. We can arrange to break up the overall integral so that every point where (**B**) fails is an end-point of the region of integration. The contributions from these pieces of the integral are usually larger than $O(t^{-1})$. How much bigger they are depends on whether higher derivatives of $\varphi(k)$ also vanish at the same place. Here we consider only the simplest case, in which $\varphi'(k)$ and $\varphi''(k)$ do not vanish simultaneously. [This is where the requirement that $F(0) = 0$ enters.] Copson (1967, p. 31) shows:

If $\varphi(k)$ has one stationary point in $K_1 \le k \le K_2$, namely at K_1, and if $\varphi''(K_1) > 0$, then as

$t \to \infty$,

$$\int_{K_1}^{K_2} Q(k)e^{i\varphi(k)t}dk = \left\{\frac{\pi}{2t\varphi''(K_1)}\right\}^{\frac{1}{2}} Q(K_1)e^{i\varphi(K_1)t+i\pi/4} + O(t^{-1}); \qquad \textbf{(D)}$$

but if $\varphi''(K_1) < 0$, then as $t \to \infty$,

$$\int_{K_1}^{K_2} Q(k)e^{i\varphi(k)t}dk = \left\{\frac{\pi}{-2t\varphi''(K_1)}\right\}^{\frac{1}{2}} Q(K_1)e^{i\varphi(K_1)-i\pi/4} + O(t^{-1}). \qquad \textbf{(E)}$$

This gives the contribution to the integral from a lower limit. The contribution from an upper limit comes from interchanging the limits and changing the sign of the integral. The total contribution from each interior stationary point of the original integral has one part from an upper limit and one from a lower limit.

Now return to the integral from problem 2. For a particular wavenumber, k^*, is there a real-valued v for which $\varphi'(k^*;v) = 0$? The definition of $\varphi(k)$ in (**A**) shows that k^* dominates the integral for that value of v for which

$$\varphi'(k^*;v) = v - \omega'(k^*) = 0, \qquad \textbf{(F)}$$

if one exists. In other words, the wave pattern that evolves from the given initial data is dominated by waves with wavenumber k^* along a straight line in $\{x,t\}$ space on which

$$\frac{x}{t} = v = \omega'(k^*) = c_g(k^*), \qquad \textbf{(G)}$$

if one exists. This shows the significance of the group velocity for any solution of the linearized water wave equations on an unbounded region.

For the linearized equations of water waves, the group velocity is an even function of k. Hence if wavenumber k^* dominates at speed v, then ($-k^*$) also dominates at that speed. Adding the contribution from both pieces gives a real-valued result for the integral, as required.

Observe from (**D**) and (**E**) that the result is an oscillatory wavetrain, with a wave envelope that decays as $t^{-\frac{1}{2}}$ (for a 2-D problem, with a 1-D water surface). The shape of the wave envelope is given by the shape of $F(k)$, the Fourier transform of the initial data. This is a general property of linearized wave evolution in a dispersive medium:
for very short times, the wave pattern looks like the shape of the initial data;
for long times, the wave envelope takes on the shape of the Fourier transform of the initial data.
It follows that if the Fourier transform has many zeroes, as $F(k)$ from problem 2 does, then the resulting wave pattern will appear as a string of wave packets, with a different wave number dominating each packet.

3. Apply the method

Find the solution of the linearized equations of water waves that evolve from the initial data given in problems 1 and 2.

a) Write down concrete formula that define $\eta(x,t)$ approximately for all x, and for large t. Make sure that your formula for $\eta(x,t)$. The initial data in problem 1 are odd in x, so their Fourier transform is odd in k. What about the solution of the problem for long times? Is it odd (in x), even or neither?

b) Sketch the graph of $\eta(x,t)$ vs. x at some large t. [There is no way to sketch a "universal" picture, valid for all large time, but a plot of $\{\eta(x,t)\sqrt{t}\}$ vs. $\{x/t\}$ comes close.] You should see wave groups. If you do, then identify the nodes of those groups (where the wave amplitude is close to zero). For large times, in what region do you find the largest wave amplitudes? What is (are) the dominant wavenumber(s) in that region?

Answers:

1. $$F(k) = \frac{8\pi^2 (aL)\sin^2(kL/2)}{i(kL)(2\pi - kL)(2\pi + kL)} = \frac{-4i\pi^2(aL)[1-\cos(kL)]}{(kL)(2\pi - k)(2\pi + k)}.$$

$F(k)$ is an odd, imaginary function of k.

$F(k) = 0$ at $kL = 2\pi n$, $n = 0, 1, 2, \ldots$

$$iF(\frac{\pi}{L}) = \frac{8aL}{3\pi},\quad iF(\frac{3\pi}{L}) = -\frac{8aL}{15\pi},\quad iF(\frac{5\pi}{L}) = -\frac{8aL}{105\pi}.$$

2. $H_-(k) = F(k)$ from problem 1, $H_+(k) = 0$,

$$\Phi\_(k) = \frac{gH\_(k)}{i\omega(k)} = -i\frac{gF(k)}{\sqrt{gk\tanh(kh)}},\ \Phi_+(k) = 0.$$

3. A copy of a correct set of graphs is shown below.