Prelude to Homework set 3: **Hamiltonian problems**

The problem of inviscid water waves can be viewed as a Hamiltonian system of partial differential equations. The simplest version of Hamiltonian machinery is for a finite-dimensional system of ordinary differential equations, with conjugate variables (p_1, q_1, p_2, q_2, …, p_N, q_N). The problems in homework set 3 are intended to point out some of the important properties of Hamiltonian systems.

Wm. Rowan Hamilton did his work in early 19th century, but the subject did not stop there. A very good, modern reference for Hamiltonian systems is V. I. Arnold's book, *Mathematical Methods of Classical Mechanics*. But Arnold requires some technical machinery that I think is not necessary in a first encounter with Hamiltonian systems. So I can't recommend a good, simpler book on the subject. Part A of what follows is a brief tour of some of the important concepts in the subject.

A. A brief tour of Hamiltonian systems (no work required by you)

1. *The prototypical problem*: For fixed $\omega^2 > 0$, find $z(t)$ so that

$$\frac{d^2z}{dt^2} + \omega^2 z = 0. \qquad (1)$$

a) A *standard method* (no Hamiltonian machinery): You have memorized that the general solution of (1) is

$$z(t) = A\sin(\omega t + \phi), \qquad (2)$$

for arbitrary (real) constants (A, ϕ). End of story.
b) A *Hamiltonian approach*: Suppose you had not memorized (2). Then multiply (1) by $\frac{dz}{dt}$ and integrate by parts to obtain

$$\frac{d}{dt}[(\frac{dz}{dt})^2 + \omega^2 z^2] = 0.$$

Conclude from this that

$$E = \frac{1}{2}[(\frac{dz}{dt})^2 + \omega^2 z^2] = \text{constant}, \qquad (3)$$

for *any* solution of (1). Several results follow from this.

- *Guess* that (1) is a Hamiltonian system, and *guess* two conjugate variables:

$$p(t) = \frac{dz}{dt}, \qquad q(t) = z(t), \qquad H(p,q) = \frac{1}{2}[p^2 + \omega^2 q^2].$$

Then verify by direct computation that Hamilton's equations,

$$\{\frac{dq}{dt}=\frac{\partial H}{\partial p},\quad \frac{dp}{dt}=-\frac{\partial H}{\partial q}\}$$

are equivalent to (1). This shows that (1) is a Hamiltonian equation, and it establishes that $\{p, q\}$ are legitimate coordinates on the phase space. As a result, arbitrary initial conditions for (1) can be given by assigning values to $\{p(0), q(0)\}$. And the solution of (1) corresponds to a curve in the $\{p,q\}$-plane, with (t) as the natural parameter to denote position along the curve. The curve starts at the point in the $\{p,q\}$ plane with coordinates $\{p(0), q(0)\}$, and t measures how far along the curve the solution has traveled from the assigned starting values.

- For $\omega^2 \geq 0$,

$$H(p,q)=\frac{1}{2}[p^2+\omega^2q^2]\geq 0, \tag{4}$$

so H can take on only non-negative values. Plot some level curves of (4) in the $\{p,q\}$ plane, one curve for each positive value of H. Each such curve is an ellipse, centered at (0,0). [The ellipse crosses the q-axis at $(\pm\sqrt{2H}, 0)$, and the p-axis at $(0, \pm\frac{\sqrt{2H}}{\omega})$. If $\omega = 1$, the ellipse is a circle with radius $\sqrt{2H}$.] Once we know that the solution cannot stop and rest somewhere along the ellipse, then it follows that every solution of (1) is periodic – the period is the time (t) required to travel around the ellipse once.

- To find the general solution of (1), and therefore to *derive* (2), solve (3) for $(\frac{dz}{dt})^2$:

$$(\frac{dz}{dt})^2=2E-\omega^2z^2,$$

so

$$\frac{dz}{dt}=\pm\sqrt{2E-\omega^2z^2},$$

and

$$\frac{\omega dz}{\sqrt{2E-\omega^2z^2}}=\pm\omega dt. \tag{5}$$

Now you're nearly done. Integrate both sides. The right side integrates to $(\omega t + \phi)$, which starts to look like (2). If you remember freshman calculus, then you recognize the integral on the left side as $\arcsin(\frac{\omega z}{\sqrt{2E}})$, or as $\sin^{-1}(\frac{\omega z}{\sqrt{2E}})$. Then taking the sine of each side gives (2). The choice of signs (±) in (5) corresponds to the upper and lower halves of the ellipse in the $\{p,q\}$ plane. OR, if you don't remember how to integrate the left side of (5), then this integral defines a function of z, so (5) defines $t = t(z)$, and its inverse defines $z(t)$, which solves (1). Done.

c) *A second example*: Consider the motion of a pendulum, with mass m, hanging from a fixed point by a rigid rod of fixed length r, and subject to a constant gravitational force, g, that acts vertically downward. (See the figure.) As the pendulum swings back and forth, the angle, θ, between the rigid rod and a vertical line satisfies

$$r \cdot \frac{d^2\theta}{dt^2} + g \cdot \sin\theta = 0. \tag{6}$$

(i) For fixed ($g > 0$, $r > 0$), find an energy integral, corresponding to (3). Call the constant of integration E.
(ii) Show that (6) is a Hamiltonian system, with a Hamiltonian proportional to E, and two conjugate variables, proportional to θ and to $\frac{d\theta}{dt}$, respectively.
(iii) For $0 \le E \le 2g/r$, sketch some level curves of E in the $\{\theta, \frac{d\theta}{dt}\}$ plane. Show that these curves are closed. But they are not ellipses.
(iv) Obtain an equation comparable to (5), which allows you to solve (6) for arbitrary initial data. It turns out that after a change of variables, that equation defines an elliptic function, which one can think of as a nonlinear generalization of sines and cosines.

d) A finite set of ordinary differential equations of even order ($2N$),

$$\frac{dz_j}{dt} = F_j(\vec{z}, t), \qquad j = 1, 2, \ldots, 2N \tag{7}$$

is said to be *Hamiltonian* if one can identify N pairs of coordinates on the phase space, $\{p_1, q_1, p_2, q_2, \ldots, p_N, q_N\}$ and a real-valued Hamiltonian function, $H(\vec{p}, \vec{q}, t)$ with the property that the set of equations in (7) are equivalent to

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}, \qquad j = 1, 2, \ldots, N. \tag{8}$$

2. *Hamiltonian machinery*
a) *Poisson brackets*: Consider a finite-dimensional Hamiltonian system, (8). Let $F(\vec{p}, \vec{q})$ be a real-valued function of the variables on the phase space. [Note that F can depend on time, but only through $(\vec{p}(t), \vec{q}(t))$.] As t changes, $p_j(t)$ and $q_j(t)$ change according to (8), so the change in F can be calculated by the chain rule:

$$\frac{dF}{dt} = \sum_{j=1}^{N} \frac{\partial F}{\partial q_j}\frac{dq_j}{dt} + \sum_{j=1}^{N} \frac{\partial F}{\partial p_j}\frac{dp_j}{dt}.$$

Using (8), we can write this as

$$\frac{dF}{dt}=\sum_{j=1}^{N}\frac{\partial F}{\partial q_j}(\frac{\partial H}{\partial p_j})+\sum_{j=1}^{N}\frac{\partial F}{\partial p_j}(-\frac{\partial H}{\partial q_j}).$$

Define [*F*, *H*], the *Poisson bracket of F and H*, to be

$$[F,H]=\sum_{j=1}^{N}(\frac{\partial F}{\partial q_j}\cdot\frac{\partial H}{\partial p_j}-\frac{\partial F}{\partial p_j}\cdot\frac{\partial H}{\partial q_j}). \qquad (9)$$

Then the time-evolution for any function on the phase space, like $F(\vec{p},\vec{q})$, that does not depend on time explicitly is given by

$$\frac{dF}{dt}=[F,H]. \qquad (10)$$

Examples:
Let $F(\vec{p},\vec{q})=p_j(t)$ for any $j \leq N$. Show that (10) implies one of the equations in (8).
Let $F(\vec{p},\vec{q})=q_j(t)$ for any $j \leq N$. Show that (10) implies another of the equations in (8).

The Poisson bracket plays a fundamental role in Hamiltonian systems.

b) *Constants of the motion*. It follows from (10) that if $G(\vec{p},\vec{q})$ is a constant of the motion of (8), then necessarily

$$[G, H] = 0. \qquad (11)$$

Eq'n (11) gives a straight-forward way to test whether a given function is a constant of the motion.

c) *Canonical transformations:* Consider first a second-order differential equation that is Hamiltonion, like (1). We chose coordinates {*p*,*q*} on the phase space, and then used them to solve (1). But a different choice of coordinates {*P, Q*} on the phase space might have worked even better. We are free to choose coordinates on the phase space, so we want the flexibility to try out various choices of coordinates, in the hopes of finding an "optimal" set. A *canonical transformation* for a second-order equation like (1) is a change of coordinates on the phase space that accomplishes two things:

- It preserves the Hamiltonian structure, so in the new coordinates {*P*,*Q*} there is a new Hamiltonian, *K*(*P*,*Q*), with the property that the equations of motion can be written in the form

$$\{\frac{dQ}{dt}=\frac{\partial K}{\partial P},\ \frac{dP}{dt}=-\frac{\partial K}{\partial Q}\}. \qquad (12)$$

- It preserves area in the phase space. For (1), this means that the Jacobian of the transformation satisfies

$$\frac{\partial(P,Q)}{\partial(p,q)} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1. \qquad (13a)$$

By using (9) with $N=1$, (13a) can also be written as

$$[P, Q] = 1. \qquad (13b)$$

For a general Hamiltonian system on a $2N$-dimensional phase space, the definition of a canonical transformation is slighlty more complicated than that in (12) and (13). We trade in one set of coordinates, $\{p_1, q_1, p_2, q_2, \ldots, p_N, q_N\}$, for another set, $\{P_1, Q_1, P_2, Q_2, \ldots, P_N, Q_N\}$. If the transformation is canonical, then the new coordinates satisfy

$$\begin{array}{c} [P_j,P_k]=0, \quad [Q_j,Q_k]=0, \\ [P_j,Q_k]=\delta_{j,k}. \end{array} \qquad j, k = 1, 2, \ldots, N. \qquad (14)$$

The last line in (14) is analogous to (13). The form of these equations provides a tidy way to identify Hamiltonian systems that can be solved completely, as we see next.

d) *Complete integrability and action-angle variables*: Given a Hamiltonian system on a $2N$-dimensional phase space, it might or might not be possible to find its general solution. If one can find the general solution of such a system, then that system is said to be *completely integrable*. The usual way to demonstrate complete integrability is to construct a canonical transformation to what are called *action-angle variables*. In these variables, $\{P_1, P_2, \ldots, P_N\}$ are called "actions", and $\{Q_1, Q_2, \ldots, Q_N\}$ are called "angles". The essential ingredient is that for a completely integrable system, the Hamiltonian depends only on the actions:

$$H = H(\vec{P}).$$

As a result, Hamilton's equations become very simple:

$$\frac{dP_j}{dt} = -\frac{\partial H}{\partial Q_j} = 0 \qquad \text{for } j = 1,2,\ldots, N,$$

so every P_j is a constant of the motion. There are N P-variables, so there are exactly N such constants. In addition,

$$\frac{dQ_j}{dt} = \frac{\partial H}{\partial P_j} \qquad \text{for } j = 1,2,\ldots, N.$$

But since $H = H(\vec{P})$, and every P_j is constant, it follows that $\frac{\partial H}{\partial P_j}$ must be constant, for $j = 1,2,\ldots, N$. Call $\frac{\partial H}{\partial P_j} = \omega_j$. Then the general solution of the system is

$$P_j = \text{constant}, \qquad Q_j = \omega_j t + Q_{j0}, \qquad j = 1,2,\ldots, N. \qquad (15)$$

The general solution must have exactly $(2N)$ arbitrary constants. They are $\{P_1, P_2, \ldots P_N; Q_{10}, Q_{20}, \ldots, Q_{N0}\}$.

Comments:

- This situation, in which $H = H(\vec{P})$ only, with no dependence on $\vec{Q}$, might seem very special. It is. Most Hamiltonian systems **cannot** be integrated completely. But every Hamiltonian system for which a complete solution is known can be put in this form.
- Every Hamiltonian system with $N = 1$ is completely integrable. In this case, we can choose

  $$P = \frac{H}{\omega},$$

  where ω is a constant to be determined. Then the equations of motion require

  $$\frac{dQ}{dt} = \omega,$$

  so

  $$Q(t) = \omega t + Q_0.$$

  The frequency, ω, can be found by requiring that $[P, Q] = 1$. If $H(P,Q)$ is positive definite, then the motion is necessarily periodic, and ω can also be found by determining the period of the motion.
- This procedure gives the general solution in terms of the transformed variables $\{\vec{P},\vec{Q}\}$, not in terms of the original variables $\{\vec{p},\vec{q}\}$. A canonical transformation is constructed to be invertible, so (in principle) the transformation can be inverted to obtain the general solution in terms of the original variables. In real life, this inversion might be a mess. As a simple example, consider that the well-known sinusoidal function in (2) can be defined to be the inverse of the function obtained by inverting the (ugly) integral in (5). To some extent, we consider "$\sin(\omega t + \phi)$" simple only because we are familiar with it.
- Counting constants of the motion is important. For a Hamiltonian system on a $2N$-dimensional phase space, it is **not** necessary to find $2N$ constants of the motion in order to construct the general solution (and establish complete integrability). It is only necessary to find N constants of the motion, provided:
  (i) each constant is independent of the other (N-1) constants; and
  (ii) every pair of constants satisfies (11).
  [Technical jargon: Then the constants are said to be "in involution".] Suppose the Hamiltonian is a constant of the motion. Then set $P_1 = H$. If one can find $(N$-1)

other constants of the motion, all independent, and if they all satisfy (11) pair-wise, then they satisfy the first set of requirements in (14), so they qualify as "actions". Then one can construct the corresponding "angles", and complete the construction of the general solution of the system. For the detailed construction, see Appendix 26 of Ergodic Problems of Classical Mechanics, by Arnold & Avez (1968).

- Consider a Hamiltonian system on a $2N$-dimensional phase space that is completely integrable. Then the N constants of the motion constrain the solution to lie on an N-dimensional manifold within the $2N$-dimensional space. If in addition this N-dimensional manifold is compact (*i.e.*, if the set is closed and bounded), then one can show that the manifold is topologically equivalent to an N-dimensional torus. For $N = 1$, this manifold is a closed curve, like the ellipses for (1) or the closed curves in (6), and the solution is necessarily periodic, like that in (2). For $N = 2$, the manifold is equivalent to the surface of a donut, with periodic motion possible in two independent directions. For $N = 3$, periodic motion is possible in 3 independent directions on the torus, *etc*. However, unless initial conditions are chosen very carefully, the trajectory of the solution will not lie in one of these special directions, and the solution will only be quasiperiodic. In this integrable case, where all solutions of the $2N$-dimensional Hamiltonian system lie on an N-dimensional, compact sub-manifold, then almost every solution of the problem comes arbitrarily close to every point on the manifold infinitely often. So the N constants of the motion constrain the solution to lie on a manifold that has only half the dimensions of the phase space, but then the trajectory of a typical solution covers the manifold densely. That describes precisely how much control one has on solutions of a completely integrable Hamiltonian system.