Homework related to lecture 3: **Hamiltonian problems**

The problem of inviscid water waves can be viewed as a Hamiltonian system of partial differential equations. The simplest version of Hamiltonian machinery is for a finite-dimensional system of ordinary differential equations, with conjugate variables (p_1, q_1, p_2, q_2, …, p_N, q_N). The problems that follow are intended to point out some of the important properties of Hamiltonian systems.

B. Problems

1. The basic problem considered in part **A** of this problem set is

$$\frac{d^2z}{dt^2} + \omega^2 z = 0. \qquad (1)$$

Consider the following variation of this problem:

$$\frac{d^2z}{dt^2} - \omega^2 z = 0, \qquad \omega^2 > 0.$$

Show that this problem is Hamiltonian. What is the Hamiltonian, and what are the conjugate variables? Do level sets of the Hamiltonian define closed curves? Are they ellipses?

2. Consider another variation of (1):

$$\frac{d^2z}{dt^2} + \omega^2 z = f(t),$$

where $\omega^2 > 0$ and $f(t)$ is a known function. Show that this equation is Hamiltonian. What is the Hamiltonian, and what are the conjugate variables? Suppose $f(t) = at + b$. Write the solution of the problem in terms of a particular solution and a homogeneous solution. Is there any simple relation between the solution of this problem and that of eq’n (1)?

3. Henon & Heiles (*Astron. Journal*, **69**, 1964) studied a system defined by the following Hamiltonian:

$$H = \frac{1}{2}[(\dot{x})^2 + (\dot{y})^2 + Ax^2 + By^2] - x^2 y - \frac{\varepsilon}{3} y^3,$$

where $\{A, B, \varepsilon\}$ are fixed constants.

a) What are the governing equations (*i.e.*, the ODEs)?

b) The Hamiltonian is written in a mixed notation, without p’s and q’s. What are proper coordinates on the phase space?

c) Consider:

$$J = x^4 + 4x^2y^2 + 4\dot{x}(x\dot{y} - y\dot{x}) - 4Ax^2y + (4A - B)(\dot{x}^2 + Ax^2)\,.$$

It has been claimed that for $\varepsilon = 6$, J is a constant of the motion. Determine whether this claim is true or false. [You can do this in at least two ways.] Justify your conclusion.
d) Suppose J turns out to be a constant. What is the significance of that (alleged) fact?

4. The linearized equations of water waves in 2-D above a flat, horizontal bottom are

$$\begin{aligned}
&\partial_t\eta = \partial_z\phi, && \\
&\partial_t\phi + g\eta = \frac{\sigma}{\rho}\partial_x^2\eta, && \text{on } z = 0, \qquad\qquad \text{(4a)} \\
&\nabla^2\phi = 0 && \text{for } -h < z < 0, \\
&\partial_z\phi = 0 && \text{on } z = -h,
\end{aligned}$$

where $\{g, h, \sigma/\rho\}$ are positive constants. You have studied aspects of these equations in earlier homework sets. Here we consider their Hamiltonian nature. We need boundary conditions in x to do that: require that the solution of (4a) be periodic in x, on $(-L < x < L)$. Then we can represent the free surface with a Fourier series,

$$\eta(x,t) = \sum_{m=-\infty}^{\infty} a_m(t)\cdot e^{im\pi x/L}\,.$$

a) The representation for $\phi(x,z,t)$ is more complicated, because of its z-dependence. Find a representation for ϕ of the form

$$\phi(x,z,t) = \sum_{m=-\infty}^{\infty} f_m(t)\cdot F_m(z)\cdot e^{im\pi x/L}\,.$$

What is $F_m(z)$?
b) The global statement from (4a) about mass conservation with periodic boundary conditions is

$$\int_{-L}^{L}[\eta(x,t)]dx = \text{constant.}$$

By adjusting where we place $z = 0$, we can require

$$\int_{-L}^{L}[\eta(x,t)]dx = 0.$$

How does this requirement simplify the representations for η and ϕ?
c) The surface elevation, η, must be real-valued, so

$$a_{-m}(t) = a_m^*(t)\,,$$

where (a_m^*) denotes the complex conjugate of a_m. What is the corresponding statement for the representation of ϕ?

d) An essential part of Zakharov's Hamiltonian formulation of the (nonlinear) equations of water waves is that the time-dependent dynamical system occurs only at the free surface. The linearized equations considered here approximate the full equations, so there should be a corresponding statement here.

1) Substitute your representations (above) into the linearized equations at $z = 0$, and show that for each (m), the two partial differential equations at $z = 0$ in (4a) lead to two, coupled, ordinary differential equations, with no coupling to other values of (m). What are these two equations?
2) Show that for each (m), these coupled equations are a Hamiltonian system. For each (m), find $\{p_m(t), q_m(t), H_m\}$ in terms of $\{a_m(t), f_m(t)$, constants$\}$.
3) (No work required by you): Observe that the PDEs in (4a) decouple into infinitely many Hamiltonian systems, each with a two-dimensional phase space.

e) Action-angle variables: You know that the linearized equations of water waves can be solved, because you used the Fourier transform representation of the solution in HW set #2.5. So the Hamiltonian system for the m^{th} Fourier mode should be transformable into action-angle variables. (You found this Hamiltonian system, in part (d) of this problem.) We know:

- The Hamiltonian is the only constant of the motion (for the m^{th} mode), so P_m must be proportional to H_m.
- For a canonical transformation, $[P_m, Q_m] = 1$.
- $Q_m(t)$ must satisfy an equation of the form $\frac{dQ_m}{dt} = \omega_m$, for some constant ω_m.
- The trajectory of the motion defined by this Hamiltonian system should be a closed orbit in the $\{p_m, q_m\}$ plane.

Based on this information, assume that $P_m(t)$ and $Q_m(t)$ have the form

$$P_m = \frac{H_m}{\omega_m}, \quad Q_m = \tan^{-1}\left(\frac{\alpha_m q_m}{p_m}\right), \tag{4b}$$

where H_m is your Hamiltonian from part (d), $\{q_m, p_m\}$ are your variables from (d), and $\{\alpha_m, \omega_m\}$are constants to be determined.

1) Substitute these forms into $[P_m, Q_m] = 1$ (where the detailed form of the Poisson bracket was spelled out in eq'n (13) of part (**A**) of this problem set), and show that the representation in (4b) is possible, but only for particular values of $\{\alpha_m, \omega_m\}$. What are these values?
2) You should recognize ω_m^2. What is it?
3) Find $P_m(t)$, $Q_m(t)$ as explicit functions of time.

Answers:

1. Let $q(t) = z(t)$, $p(t) = \frac{dz}{dt}(t)$. Then $H = \frac{1}{2}[p^2 - \omega^2 q^2]$, and the equation is Hamiltonian. The level sets are hyperbolae. They are **not** closed curves.

2. (a) Again, let $q(t) = z(t)$, $p(t) = \frac{dz}{dt}(t)$. Then $H = \frac{1}{2}[p^2 + \omega^2 q^2] - q \cdot f$, and the equation is Hamiltonian. But the Hamiltonian is *not* a constant of the motion.

(b) If $f(t) = at + b$, then a particular solution is $z_p(t) = \frac{at+b}{\omega^2}$, and the homogeneous solution is the solution of eq'n (1). A break-up like this occurs valid for **any** piecewise smooth $f(t)$. The particular solution has no variation, and the homogeneous solution is the solution of (1).

3. (a) $\ddot{x} + Ax - 2xy = 0, \quad \ddot{y} + By - x^2 - \varepsilon y^2 = 0$.

(b) Several answers are possible. One is $\{q_1 = x, \quad q_2 = y, \quad p_1 = \dot{x}, \quad p_2 = \dot{y}\}$, so

$$H = \frac{1}{2}[(p_1)^2 + (p_2)^2 + Aq_1{}^2 + Bq_2{}^2] - q_1{}^2 q_2 - \frac{\varepsilon}{3} q_2{}^3.$$

(c) $J = q_1{}^4 + 4q_1{}^2 q_2{}^2 + 4p_1(p_1 q_2 - p_2 q_1) - 4Aq_1{}^2 q_2 + (4A - B)(p_1{}^2 + Aq_1{}^2)$, so

$$\frac{\partial J}{\partial q_1} = 4q_1{}^3 + 8q_1 q_2{}^2 - 4p_1 p_2 - 8Aq_1 q_2 + 2A(4A - B)q_1,$$

$$\frac{\partial J}{\partial q_2} = 8q_1{}^2 q_2 + 4p_1{}^2 - 4Aq_1{}^2,$$

$$\frac{\partial J}{\partial p_1} = 8p_1 q_2 - 4p_2 q_1 + 2(4A - B)p_1, \quad \frac{\partial J}{\partial p_2} = -4p_1 q_1.$$

Then show that $\frac{dJ}{dt} = [J, H] = 0$.

(d) Some combination of J and H provides the "actions" of action-angle variables – the problem is completely integrable.

4.(a) $F_m(z) = K_m \cosh(\frac{m\pi(z+h)}{L})$. (b) $a_0 = 0$, $f_0 = \text{const } (= 0)$.

(c) $f_{-m}(\mathrm{t}) = f_m{}^*(t)$.

(d1) Depending on K_m from (a),

$$\dot{a}_m = (\frac{m\pi}{L})\sinh(\frac{m\pi h}{L}) f_m,$$
$$\dot{f}_m \cosh(\frac{m\pi h}{L}) + (g + \frac{\sigma}{\rho}(\frac{m\pi}{L})^2) a_m = 0.$$

(d2) Again, several answers are possible. One is: $p_m(t) = f_m(t)$, $q_m(t) = a_m(t)$,

$$H_m = \frac{1}{2}[(\frac{m\pi}{L})\sinh(\frac{m\pi h}{L}) p_m{}^2 + \frac{g + \frac{\sigma}{\rho}(\frac{m\pi}{L})^2}{\cosh(\frac{m\pi h}{L})} q_m{}^2].$$

(e) Find $[P_m, Q_m] = \frac{\alpha_m}{\omega_m}\{\frac{2H_m}{p_m{}^2 + (\alpha_m q)^2}\}$.

Then $[P_m, Q_m] = 1$

$$\Rightarrow \quad \alpha_m{}^2 = \frac{g + \frac{\sigma}{\rho}(\frac{m\pi}{L})^2}{(\frac{m\pi}{L})\sinh(\frac{m\pi h}{L})\cosh(\frac{m\pi h}{L})}$$

$$\Rightarrow \quad \omega_m{}^2 = (\frac{m\pi}{L})\{g + \frac{\sigma}{\rho}(\frac{m\pi}{L})^2\}\tanh(\frac{m\pi h}{L}).$$