Well-posedness of the water-wave equations

Lecture 9

A. Definitions (Hadamard, 1908):

1. A system of differential equations, along with its initial &/or boundary conditions, is <u>well posed</u> if
   1) a solution exists;
   2) the solution is unique; and
   3) the solution depends continuously on the boundary &/or initial data.

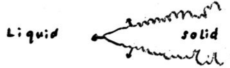
2. If such a system of differential equations is not well posed, then it is <u>ill posed</u>

3. Global well-posedness vs. well posed for some finite time.

Q: Is well-posedness of interest only to purists?

Won't a good numerical code exhibit any problems of ill-posedness?

Counter-example:
Growth of a dendritic crystal in a super cooled liquid:



Kruskal & Segur (1991) showed that the "geometric model" has no solutions.

Well posedness of water-wave equations

1. Early work on initial-value problems:
   a) Restricted mostly to 2-D flows, so free surface is a 1-D curve.
   b) Considered only small deviations from a flat surface at $t=0$

2. Recent work (since 1997)
   a) Sijue Wu (1997)
   - 2-D flows on deep water, with no surface tension
   - She proves well-posedness for some finite time in Sobolev spaces, given arbitrary initial cond. in Sobolev sp.

2. Recent work

b) Wu (1999)

- 3-D flows on deep water, with no surface tension
- She proves well posedness for finite time in Sobolev spaces, as before

c) Lannes (2005)

- 3-D flows on water of finite depth, with no surface tension
- Depth of quiescent depth, $h(x, y)$, is arbitrary, but for one constraint
- Uses Zakharov's variables, $\eta(x, y, t)$ & $\psi(x, y, t)$
- Proves well-posedness in Sobolev spaces

2. Recent work

d) Coutand & Shkoller

- 3-D flows
- with or without surface tension
  (2 proofs)
- not restricted to irrotational flows
- Prove well posedness for finite time in Sobolev spaces with optimal regularity

How does Lannes' work relate to earlier work by Zakharov?

1. Recall Z's formulation:

a) $\eta(x,y,t)$: location of free surface

$\psi(x,y,t)$: velocity potential at $z=\eta$

b) 2 evolution equations on $z=\eta$:

$$\partial_t \eta = \sqrt{1+|\nabla\eta|^2}\ \partial_n \phi\Big|_{z=\eta}$$

$$\partial_t \psi = \cdots$$

c) Dirichlet-to-Neumann map:

$$\partial_n \phi\Big|_{z=\eta} = \iint G(x,y;\mu,\nu)\, \psi(\mu,\nu,t)\, d\mu\, d\nu$$

d) Zakharov used fact that G exists & behaves tolerably.

2. Lannes' formulation

a) Also uses Dirichlet-to-Neumann map

$$\partial_n \phi\big|_{z=\eta} = \iint G(x,y;\mu,\nu)\, \psi(\mu,\nu,t)\, d\mu\, d\nu$$

b) Important technical result by Lannes:

· Define $\hat{G}$ by

$$\partial_t \eta = \hat{G}\psi = \sqrt{1+|\nabla\eta|^2}\; \partial_n \phi = \sqrt{1+|\nabla\eta|^2} \iint G\psi\, d\mu\, d\nu$$

$$\cdot\ \|\hat{G}\psi\|_{H^{k+\frac12}} \le C(k, |\eta|_{H^{s_0}}) \Big[|\eta|_{H^{k+3/2}} |\nabla\psi|_{H^{s_0-1}} + |\nabla\psi|_{H^{k+\frac12}} \Big]$$

By controlling $\hat{G}$, Lannes controls $\partial_t\eta$ & $\partial_t\psi$, & proves well-posedness